# B.A./B.Sc. 5th Semester (Honours) Examination, 2023 (CBCS)

## Subject : Mathematics

## Course : BMH5CC11

## (Partial Differential Equations and Applications)

**Time: 3 Hours** **Full Marks: 60**

*The figures in the margin indicate full marks.*
*Candidates are required to give their answers in their own words as far as practicable.*
*Notation and symbols bear usual meaning.*

**1.** Answer *any ten* questions: 2×10=20

(a) Find the differential equation of the set of all right circular cones whose axes coincide with $z$-axis.

(b) Define order and degree of a partial differential equation with example.

(c) Solve the partial differential equation $u_x^2 + u_y^2 = u$ using $u(x, y) = f(x) + g(x)$.

(d) Find the partial differential equation of the family of planes, the sum of whose $x, y, z$ intercepts is equal to unity.

(e) Classify the following partial differential equations with proper reason whether they all linear, non-linear, semi-linear or quasi-linear: 1+1

(i) $xzp + x^2yz^2q = xy$

(ii) $xyp + x^2yq = x^2y^2z^2$

(f) Find the characteristic curve for the equation $x\frac{\partial u}{\partial y} - y\frac{\partial u}{\partial x} = u$ in $x$-$y$ plane.

(g) Find the general integral of $\frac{y^2z}{x}p + xzq = y^2$.

(h) Determine the region where the given partial differential equation $yu_{xx} + xu_{yy} = 0$ is hyperbolic in nature.

(i) Changing the independent variables by taking $u = y - x$ and $v = \frac{1}{2}(y^2 - x^2)$, find the value of $\frac{\partial^2 z}{\partial x\,\partial y}$.

(j) Find the family of surfaces orthogonal to the family of surfaces whose PDE is $(y + z)p + (z + x)q = x + y; p = \frac{\partial z}{\partial x},\ q = \frac{\partial z}{\partial y}$.

(k) Show that $u(x, t) = \phi(x + ct) + \psi(x - ct)$ is a solution of the equation $c^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$, where $\phi$ and $\psi$ are arbitrary functions.

(l) If $u = x\sin^{-1}\frac{y}{x} + y\tan^{-1}\frac{x}{y}$, then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ is $\frac{3\pi}{4}$ at (1, 1).

(m) Write down one-dimensional heat equation and indicate its nature.

(n) Solve $x^3\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} = 0$ using method of separation of variables if $u(0, y) = 10\,e^{5/y}$.

(o) Solve: $y^2(x-y)p + x^2(y-x)q = z(x^2+y^2)$

**2.** Answer *any four* questions: 5×4=20

(a) Find the integral surface of the differential equation $2y(z-3)p + (2x-z)q = y(2x-3)$ which passes through the circle $z = 0,\ x^2+y^2 = 2x$.

(b) Prove that the general solution of the semi linear partial differential equation $Pp + Qq = R$ is $F(u, v) = 0$ where $u$ and $v$ such that $u = u(x, y, z) = c_1$ and $v = v(x, y, z) = c_2$ are solution of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ [$c_1, c_2$ are constants].

(c) Solve the Cauchy problem by method of characteristics $p - zq + z = 0$, for all $y$ and $x > 0$ for the initial data curve $c: x_0 = 0, y_0 = t, z_0 = -2t, -\infty < t < \infty$.

(d) Reduce the equation $\frac{\partial^2 z}{\partial x^2} = (1+y)^2\frac{\partial^2 z}{\partial y^2}$ to canonical form.

(e) Solve the partial differential equation by the method of separation of variables:

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$$

(f) Find a complete integral of $x(1+y)p = y(1+x)q$.

**3.** Answer *any two* questions: 10×2=20

(a) (i) Consider partial differential equation of the form $ar + bs + ct + f(x, y, z, p, q) = 0$ in usual notation, where $a, b, c$ are constants. Show how the equation can be transformed into its canonical form when $b^2 - 4ac = 0$.

(ii) Solve: $p + 3q = z + \cot(y - 3x)$ 6+4

(b) (i) Reduce the following to canonical form and hence solve:

$$x^2 r + 2xys + y^2 t = 0 \ \left(r \equiv \frac{\partial^2 z}{\partial x^2}, s \equiv \frac{\partial^2 z}{\partial x \partial y}, t \equiv \frac{\partial^2 z}{\partial y^2}\right)$$

(ii) Find the characteristics of $\frac{\partial^2 z}{\partial x^2} + y^2\frac{\partial^2 z}{\partial y^2} = y$. (5+2)+3

(c) (i) Obtain D'Alembert's solution of following Cauchy problem of an infinite string:

$$u_{tt} - c^2 u_{xx} = 0,\ x \in \mathbb{R}, t > 0$$
$$u(x, 0) = f(x)$$
$$u_t(x, 0) = g(x)\ \forall\, x \in \mathbb{R}$$

(ii) Solve the following problem by method of characteristics: 5+5

$$z_x + z z_y = 1$$
$$z(0, y) = ay,\ a = \text{const.}$$

(d) (i) Use the method of separation of variable to solve the equation $\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t}$, given that $v = 0$ when $t \to \infty$ as well as $v = 0$ at $x = 0$ and $x = l$.

(ii) Verify that $z = f(y + ix) + g(y - ix) - (m^2 + n^2)^{-1}$ is a solution of $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \cos mx \, . \cos ny$. 5+5

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